Sidon sets

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1 anti-Sidon size of $\mathbf{Z}_{37}$

Problem

Show that every 10 element subset B of $A = \{1, 2, \ldots, 37\}$ contains four pairwise distinct elements, such that the sum of some two equals the sum of the remaining two.

We prove the claim when x, y, z, t are required to be pairwise distinct. Observe that if distinct integers $x, y, z, t \in B$ satisfy the sum condition

$$x + y = z + t,$$

we may without loss of generality assume $x > y$, $z > t$ and $x > z$. If it were that $y > z$, the left side of the equality would be larger than the right, hence it must be that $z > y$ and by a similar argument $t > y$. It follows that $x > z > t > y$, from which it follows that any two pairs that satisfy the sum condition also satisfy the following difference condition

$$x - t = z - y > 0.$$

There are $\binom{10}{2} = 45$ pairs of distinct elements of B, each of which forms one positive difference. These differences take values from the 36-element set $C = \{1, 2, \ldots, 36\}$. If $(a, b) \in B \times B$ is a pair of distinct elements, we will say that the value $|a - b| \in C$ is *represented* by it.

By the pigeonhole principle it follows that some elements of C are represented twice. If any element of C is represented three times it follows easily that at least two of the pairs have pairwise different elements, meaning the sum condition is satisfied. Assume this does not occur.

By the pigeonhole principle it follows that at least 9 elements of C are represented twice. If an element of C is represented by two pairs that contain four distinct elements, the sum condition is easily seen to be satisfied. Assume that none of the representations are formed by four distinct integers.

Since none of the double representations are formed by four distinct elements of B, yet the double-representing pairs are distinct, it follows that the 9 double representations are formed by pairs of the form

$$(c_j - r_j, c_j) \quad \text{and} \quad (c_j, c_j + r_j)$$

for $1 \leq j \leq 9$ and positive integers $c_1, \ldots, c_9$ and $r_1, ..., r_9$ of appropriate size.

If it were that $c_i = c_j$ for some $i \neq j$, then the sum condition would be satisfied by the clearly pairwise distinct integers

$$(c_i - r_i) + (c_i + r_i) = (c_j - r_j) + (c_j + r_j).$$

Assume that this doesn't occur, hence $\{c_i\}$ are pairwise distinct. However, B contains both a smallest and a largest element, which can't be among the $\{c_i\}$, as there exist both larger and smaller elements than any of the $\{c_i\}$. By yet another application of the pigeonhole principle it follows that one of the non-extremal elements of B must appear among the $\{c_i\}$ twice, which is contradictory. It follows that all 10 element subsets of A contain four pairwise distinct elements that satisfy the sum condition.

2 Generalization

Fix $N \in \mathbb{N}$ and replace A with $\{1, 2, \ldots, N\}$. We are now interested in how small k can be, if we still want all k element subsets of $\{1, 2, \ldots, N\}$ to be *aSidon*, which we define as follows:

A a subset $B \subseteq A$ is *aSidon* if all sums of two distinct elements are distnct. The smallest integer k, for which it holds that all k-element subsets of A are aSidon is called the *aSidon* size of A.

In the general case there exist $\binom{k}{2}$ pairs of distinct elements, their positive differences taking values from the $N-1$ element set $\{1, 2, \ldots, N-1\}$. Since we want more than $k-2$ distinct 3-term aritmetic progressions to prove it is aSidon by the argument described above, we desire the inequality:

$$\binom{k}{2} > (k-2) + (N-1).$$

By rearranging the inequality above, it follows that if a subset of $\{1, 2, \ldots, N\}$ contains

$$k = \left\lceil \sqrt{2N - \frac{15}{4}} + \frac{3}{2} \right\rceil$$

elements, then it must be aSidon.

References

[1] Po-Shen Loh. *Pidgeonhole Principal*. 2010. URL: https://www.math.cmu.edu/~ploh/docs/math/mop2010/pigeonhole.pdf (visited on 05/04/2024).